Non-Classical Metatheory from Above

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... in particular ones which still resort to using classical logic in the metametalanguage.

The unacceptability of using Classical Logic in the Metalangauge

"How far can a logician who professes to hold that perfectionism is the correct criterion of valid argument, but who freely accepts and offers standard mathematical proofs, in particular for theorems about perfectionist logic itself, be regarded as sincere or serious in objecting to classical logic?"

—J.P Burgess, 'No Requirement of Relevance', in S. Shapiro (Ed.) Oxford Handbook of Philosophy of Mathematics and Logic, OUP (2005), p.740

Justifying this practice

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> *"The "worlds semantics" arose in response to the claim that, for non-classical systems of logic to be "intelligible", a truth-functional semantics was required. ... [T]he aim was to "preach to the Gentiles in their own tongue"."*
>
> —R.K. Meyer,
>
> *'Proving Semantical Completeness "Relevantly" for R', Logic Research Paper (23), RSSS Australian National University (1985), p.1*

Pushing Classical Logic One Rung Up

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- We might hope, then, to be able to prove these results about L within classical models of L-set-theory.
- So reasoning about L using L in the *metalanguage*, but classical logic in the *metametalanguage*.
- In [Bacon, 2013] this is done using algebraic models for a range of non-classical logics, including those which don't have well behaved proof-theoretic presentations.

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  - Our Case: Andrew Bacon's non-classical soundness and completeness results conducted in classical models of non-classical set-theory.
  - Our Logic: Intuitionisitc Propositional Logic.
- Ultimately I think this casts some doubt on the helpfulness of Bacon's results for the cases he cares about where it's unclear what fully using L in the metalangauge might look like.

Plan

The Intuitionistic Theory of Species

(Model) Internal Interpretations

(Reflexive) Internal Completeness

Throwing Away The Ladder?

The Intuitionistic Theory of Species

Refresher: Propositional Kripke Models

- A Kripke model for intuitionistic logic is a structure $\mathcal{M} = \langle K, \leqslant, V \rangle$ where K is a set, $\leqslant$ is a reflexive and transitive relation on K, and V is a function from propositional variables to subsets of K closed under persistence:

$$(\textit{Persistence}) \quad \text{If } a \in V(p_i) \text{ and } a \leqslant b, \text{ then } b \in V(p_i)$$

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- Truth at a world in a model is defined recursively:
    - $\mathcal{M} \models_a p_i$ iff $a \in V(p_i)$
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  - $\mathcal{M} \models_a A \rightarrow B$ iff $\forall b$ if $a \leqslant b$, then if $\mathcal{M} \models_b A$ then $\mathcal{M} \models_b B$.

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  - $\mathcal{M} \models_a \neg A$ iff $\forall b$ if $a \leqslant b$, then $\mathcal{M}, b \not\models A$.
  - $\mathcal{M} \models_a A \to B$ iff $\forall b$ if $a \leqslant b$, then if $\mathcal{M} \models_b A$ then $\mathcal{M} \models_b B$.
- In the first-order case things often get slightly complicated, but we will elide over these issues by only considering constant domain models, in which case our quantifiers can be interpreted in the familiar (classical) way.

We'll work with Models of the Intuitionistic Theory of Species, following [de Jongh and Smorynski, 1976].

Definition

A model of the intuitionistic theory of species is a structure $\langle K, \leqslant, D_1, D_2 \rangle$ *where* K *is a set,* $\leqslant$ *a partial order on* K*,* $D_1 = \omega$ *and* D_2 *is the set of all systems of sets, where* X *is a system of sets iff (i)* $X = \{T_a | a \in K \text{ and } T_a \subseteq \omega\}$ *and (ii) if* T_a *and* T_b *are in* X*, and* $a \leqslant b$ *then* $T_a \subseteq T_b$*.*

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We can then interpret claims about species at a world a in a model $\mathcal{M}$ relative to a variable assignment v by letting

$$\langle K, \leqslant, D_1, D_2 \rangle, v \models_a t \in T \iff v(t) \in v(T)(a)$$

where $v(T) \in D_2$.

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where $v(T) \in D_2$.

We'll assume that our models all validate Heyting Arithemtic so we can do Gödel encoding, etc. (Bacon does this by fiat)

(Model) Internal Interpretations

"We can give an interpretation of one or more formulas of ICP by specifying some inhabited species $\underline{D}$ (a species which we can show to have at least one element) as the domain of the individual variables, and assigning to each individual constant an element of $\underline{D}$ and to each n-place predicate-letter a subspecies of $\underline{D}^n$. Without attempting to give any non-circular explanations of the logical constants, but simply taking their intuitionistic meanings for granted, we are then entitled to assume that, from an intuitionistic standpoint, to say that a formula comes out true under an interpretation of this kind has a perfectly determinate content. Let us call an interpretation of this sort an internal interpretation."

—*Michael Dummett, Elements of Intuitionism Second Edition, OUP (2000), p.155*

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- The connectives are then interpreted homophonically, so, for example, we say that a conjunction is true relative to an internal interpretation just in case both its conjuncts are true relative to that internal interpretation.
- What we'll now do is show how we can construct an internal interpretation of intuitionistic logic within a species model, using the same general strategy as that used in [Bacon, 2013].

Models of IL in models of species

Say that a species X is a *model* iff

$$\forall x(x \in X \to Atomic(x))$$

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$$\forall x(x \in X \rightarrow Atomic(x))$$

Then given X a species, and $\ulcorner \cdot \urcorner$ an encoding of formulas of propositional IL as numerals define the predicate $Tr(X, x)$ inductively as:

- $Tr(X, \ulcorner p_i \urcorner)$ iff $\ulcorner p_i \urcorner \in X$.
- $Tr(X, \ulcorner A \wedge B \urcorner)$ iff $Tr(X, \ulcorner A \urcorner) \wedge Tr(X, \ulcorner B \urcorner)$
- $Tr(X, \ulcorner A \vee B \urcorner)$ iff $Tr(X, \ulcorner A \urcorner) \vee Tr(X, \ulcorner B \urcorner)$
- $Tr(X, \ulcorner \neg A \urcorner)$ iff $\neg Tr(X, \ulcorner A \urcorner)$
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We can then formalize the claim that A is valid (=$Valid(\ulcorner A \urcorner)$) as $\forall X(Model(X) \to Tr(X, \ulcorner A \urcorner))$

Internal Interpretations Internalize Kripke Models

Proposition

Suppose that $\langle K, \leqslant, D_1, D_2 \rangle, v \models_a Model(X)$. *Then letting* $V_X(p_i) = \{b \in K | \langle K, \leqslant, D_1, D_2 \rangle, v \models_b \ulcorner p_i \urcorner \in X\}$ *it follows that* $\langle K, \leqslant, V_X \rangle$ *is a propositional Kripke model s.t.*

$$\langle K, \leqslant, V_X \rangle \models_a A \text{ if and only if } \langle K, \leqslant, D_1, D_2 \rangle, v \models_a Tr(X, \ulcorner A \urcorner)$$

- Persistence for V_X follows directly from the persistence of species membership.
- Otherwise the proof is relatively direct given the fact that the *Tr*-conditions are homophonic.

Validity in a Species model is Validity on a Frame

Theorem

Suppose that $\langle K, \leqslant, D_1, D_2 \rangle$ is an intuitionsitic model of the species. For all formulas A, and all $a \in K$ we have the following:

$$\langle K, \leqslant, D_1, D_2 \rangle \models_a \mathit{Valid}(\ulcorner A \urcorner) \text{ if and only if } \forall V : \langle K, \leqslant, V \rangle \models_a A$$

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- The 'only if' direction here follows from the previous lemma and the fact that for any $\langle K, \leqslant \rangle$-valuation V, X_V is a species which contains only codes of atomic formulas (and so satisfies $\mathit{Model}(X)$).

$$X_V = \{T_a | a \in K \,\&\, T_a = \{\ulcorner p_i \urcorner | a \in V(p_i)\}\}$$

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$$X_V = \{T_a | a \in K \,\&\, T_a = \{\ulcorner p_i \urcorner | a \in V(p_i)\}\}$$

- So given a valuation V, we can form X_V which will satisfy the hypothesis of the previous lemma, and then appeal to the 'if' direction to conclude that $\langle K, \leqslant, V\rangle \models_a A$.

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- For the 'if' direction suppose that we have $\langle K, \leqslant, D_1, D_2\rangle v, \models_b \mathit{Model}(X)$. Then by the persistence and our hypothesis we have $\langle K, \leqslant, V_X\rangle \models_b A$.

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- So by the lemma we have $\langle K, \leqslant, D_1, D_2\rangle, v \models_b \mathit{Tr}(X, \ulcorner A \urcorner)$
- But b and X were arbitrary so we have $\langle K, \leqslant, D_1, D_2\rangle, v \models_a \mathit{Valid}(\ulcorner A \urcorner)$

Let $\mathcal{T}_s = \langle K_{\mathcal{T}}, \leqslant_{\mathcal{T}}, D_1, D_{2\mathcal{T}} \rangle$, where $\mathcal{T} = \langle K_{\mathcal{T}}, \leqslant_{\mathcal{T}} \rangle$ is the full countable binary tree. So by the previous result we have:

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In particular, given that (as shown by [Kirk, 1979]), the frame $\mathcal{T}$ is characteristic for intuitionistic propositional logic, this means that:

Theorem

(i) $\mathcal{T}_s \models_a \mathit{Valid}(\ulcorner A \urcorner)$ *if and only if* $\mathcal{T}_s \models_a \mathit{Prov}(\ulcorner A \urcorner)$
(ii) $\mathcal{T}_s \models_a \forall \ulcorner A \urcorner (\mathit{Valid}(\ulcorner A \urcorner) \leftrightarrow \mathit{Prov}(\ulcorner A \urcorner))$

Where $\mathit{Prov}(\ulcorner A \urcorner)$ is the (codes of) all those formulas A which are valid in all propositional Kripke models.

- So what we have here is a proof of Soundness and Completeness for Intuitionistic Propositional Logic conducted within a model of intuitionistic set theory.

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- Additionally throughout we were working within a *classical* model of an *intuitionistic* theory.
- So our question now becomes: can we recover these results—-soundness & completeness for intuitionistic propositional logic w.r.t. internal interpretations—in a fully intuitionistic setting?

(Reflexive) Internal Completeness

Reflexive Internal Completeness

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- In this language a *truth function* M is a function from (codes of) propositional formulas to $\wp(\{0\})$ where (writing $M \models \varphi$ for $0 \in M(\ulcorner\varphi\urcorner)$) we have:
    1. $M \models \neg\varphi \iff M \not\models \varphi$
    2. $M \models \varphi \wedge \psi \iff M \models \varphi \wedge M \models \psi$
    3. $M \models \varphi \vee \psi \iff M \models \varphi \vee M \models \psi$
    4. $M \models \varphi \to \psi \iff M \models \varphi \to M \models \psi$
    5. $M \not\models \bot$

- To work withing a resolutely intuitionistic metatheory we'll follow [Carter, 2006] in working in second-order Heyting arithmetic (HAS).
- In this language a *truth function* M is a function from (codes of) propositional formulas to $\wp(\{0\})$ where (writing $M \models \varphi$ for $0 \in M(\ulcorner\varphi\urcorner)$) we have:
    1. $M \models \neg\varphi \iff M \not\models \varphi$
    2. $M \models \varphi \wedge \psi \iff M \models \varphi \wedge M \models \psi$
    3. $M \models \varphi \vee \psi \iff M \models \varphi \vee M \models \psi$
    4. $M \models \varphi \to \psi \iff M \models \varphi \to M \models \psi$
    5. $M \not\models \bot$
- Given an intermediate propositional logic L let $Meta(L)$ be HAS extended with the second-order instances of the axioms of L.

The Model-internal ladder can't be thrown away

To be able to throw away the Model-internal ladder we'd have to be able to show that:

$$Meta(IL) \vdash \forall \varphi (\forall M (M \models \varphi) \rightarrow Prov_{IL}(\varphi))$$

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As it happens, though, this is not the case:

Theorem ([Carter, 2006])
If L is an intermediate propositional logic such that $Meta(L) \vdash \forall\varphi(\forall M(M \models \varphi) \rightarrow Prov_{L}(\varphi))$ then L is classical logic.

Why? Suppose $Meta(L) \vdash \forall M(M \models \varphi) \rightarrow Prov_{L}(\varphi)$. Then we know that $Meta(\mathrm{CL}) \vdash \forall M(M \models \varphi) \rightarrow Prov_{L}(\varphi)$, but also $Meta(\mathrm{CL}) \vdash \forall M(M \models \varphi) \leftrightarrow Prov_{\mathrm{CL}}(\varphi)$. So by the transitivity of $\rightarrow$ we have $Meta(\mathrm{CL}) \vdash Prov_{\mathrm{CL}}(A) \leftrightarrow Prov_{L}(A)$. But we can show that that $\vdash_{CL} A$ iff $Meta(CL) \vdash Prov_{\mathrm{CL}}(A)$ iff $Meta(CL) \vdash Prov_{L}(A)$ iff $\vdash_{L} A$, so $L = CL$, as claimed.

Digression: Internal Completeness in Intuitionistic Logic

As it happens there are a number of different distinct notions of completeness in Intuitionistic metatheory.

- **Strong Formula Completeness** ($\forall M(M \models \varphi) \rightarrow \vdash \varphi)$)which we've just seen that intuitionistic logic lacks.
- **Weak Formula Completeness** ($\nvdash \varphi \rightarrow \exists M(M \not\models \varphi)$) which [Dummett, 1977] shows *IL* has w.r.t. finite Kripke models.
- **Countermodel Completeness** ($\Gamma \nvdash \bot \rightarrow \exists M(M \models \Gamma)$) which [Carter, 2006] shows can only be had by extensions of $IL + \neg\neg\psi \vee \neg\psi$

Throwing Away The Ladder?

A Moral

- If model-internal completeness proofs are to vindicate non-classical logics, we would need to have some hope that we could eventually throw away the ladder.
- I think the results above show that in general this is to hope in vain.
- In the intuitionistic case this is, perhaps, to be expected given the Browerian point (echoed by D.C. McCarty) that 'logic must be ancillary to mathematics'[1]
- For the cases Bacon himself is more interested in concerning logics for vagueness, the outlook is perhaps slightly more grim.

[1] D.C. McCarty. 'Intuitionism in Mathematics' in S. Shapiro (Ed.) 'Oxford Handbook of Philosophy of Mathematics and Logic', OUP (2005), p.373

Thank You!
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References I

Bacon, A. (2013).
Non-classical Metatheory for Non-classical Logics.
Journal of Philosophical Logic, 42:335–355.

Carter, N. C. (2006).
Reflexive Intermediate Propositional Logics.
Notre Dame Journal of Formal Logic, 47(1):39–62.

de Jongh, D. and Smorynski, C. (1976).
Kripke Models and the Intuitionistic Theory of Species.
Annals of Mathematical Logic, 9:157–186.

Dummett, M. A. E. (1977).
Elements of Intuitionism.
Oxford University Press, Oxford, (second edition, 2000) edition.

References II

Kirk, R. E. (1979).
Some Classes of Kripke Frames Characteristic For the Intuitionistic Logic.
Zeitschr. f. math. Logik und Grundlagen d. Math., 25:409–410.